

A NOTE ON THE CONCORDANCE INVARIANTS EPSILON AND UPSILON

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ABSTRACT. Ozsváth-Stipsicz-Szabó [OSS14] recently defined a one-parameter family $\Upsilon_K(t)$ of concordance invariants associated to the knot Floer complex. We compare their invariant to the $\{-1, 0, 1\}$ -valued concordance invariant $\varepsilon(K)$, which is also associated to the knot Floer complex. In particular, we give an example of a knot K with $\Upsilon_K(t) \equiv 0$ but $\varepsilon(K) \neq 0$.

1. INTRODUCTION

Beginning with the \mathbb{Z} -valued concordance homomorphism $\tau(K)$ [OS03], the knot Floer homology package [OS04, Ras03] has yielded an abundance of concordance invariants. One of the benefits of these invariants, as opposed to classical concordance invariants such as signature, is that they can be non-vanishing on topologically slice knots. For example, we have the following theorem.

Theorem 1 ([Hom13, Theorem 1]). *The subgroup of the smooth concordance group given by topologically slice knots contains a direct summand isomorphic to \mathbb{Z}^∞ .*

The proof of the above theorem relies on the $\{-1, 0, 1\}$ -valued concordance invariant $\varepsilon(K)$ associated to the knot Floer complex [Hom11a, Definition 3.1]. The quotient of the concordance group by the subgroup $\{K \mid \varepsilon(K) = 0\}$ is totally ordered, and properties of the order structure can be used to construct linearly independent concordance homomorphisms.

Ozsváth-Stipsicz-Szabó [OSS14, Theorem 1.20] recently gave a new proof of Theorem 1, using a one-parameter family $\Upsilon_K(t)$ of \mathbb{R} -valued concordance homomorphisms also associated to the knot Floer complex. Both ε and Υ are strictly stronger than τ in that

$$\varepsilon(K) = 0 \text{ implies } \tau(K) = 0 \quad \text{and} \quad \Upsilon_K(t) \equiv 0 \text{ implies } \tau(K) = 0,$$

but there exist knots K with $\tau(K) = 0$ while $\varepsilon(K) \neq 0$ and $\Upsilon_K(t) \not\equiv 0$. One such example is the knot $T_{3,4} \# -T_{2,7}$, where $T_{p,q}$ denotes the (p, q) -torus knot and $-K$ denotes the reverse of the mirror image of K .

The knot Floer complex $CFK^\infty(K)$ is a bifiltered chain complex associated to the knot K . We call the two filtrations the vertical and horizontal filtrations. The invariants ε and Υ are both defined using the bifiltration, while the definition of τ uses only one of the two filtrations. Roughly, $\varepsilon(K)$ is a measure of how the vertical filtration interacts with the horizontal filtration: the so-called vertical homology has rank one, and ε measures whether this homology class is a boundary, cycle, or neither in the horizontal homology. On the other hand, the idea behind $\Upsilon_K(t)$ is to apply a linear transformation to the bifiltration on the knot Floer complex and then look at the grading of a certain distinguished generator in the homology of the resulting complex.

More generally, both ε and Υ are invariants of not just knots, but of (suitable) bifiltered chain complexes. In [OSS14, Proposition 9.4], Ozsváth-Stipsicz-Szabó give an example of a complex C with $\varepsilon(C) = 0$ but $\Upsilon_C(t) \not\equiv 0$, although it is currently unknown if the complex C is realized as CFK^∞ of a knot. Conversely, we prove the following.

Theorem 2. *There exist knots K with $\Upsilon_K(t) \equiv 0$ but $\varepsilon(K) \neq 0$.*

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The knots used in the above theorem are connected sums of certain (iterated) torus knots.

An interesting question to consider is what obstructions to sliceness can be extracted from $CFK^\infty(K)$ when $\Upsilon_K(t) \equiv 0$ and $\varepsilon(K) = 0$.

Recall that the *concordance genus* of K , $g_c(K)$, is the minimal Seifert genus of any knot K' which is concordant to K . The function $\Upsilon_K(t)$ is a piecewise-linear function of t whose slope has finitely many discontinuities [OSS14, Proposition 1.4]. Let s denote the maximum of the finitely many slopes appearing in the graph of $\Upsilon_K(t)$. Ozsváth-Stipsicz-Szabó [OSS14, Theorem 1.13] prove that

$$s \leq g_c(K).$$

There is also a concordance genus bound $\gamma(K)$, defined using ε [Hom12].

Corollary 3. *There exist knots K for which the concordance genus bound given by $\Upsilon_K(t)$ is zero, but $\gamma(K) \neq 0$.*

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2. THE EXAMPLE

We will let $T_{p,q;s,t}$ denote the (s,t) -cable of $T_{p,q}$, where s denotes the longitudinal winding. We assume the reader is familiar with the knot Floer complex; see, for example, [Hom11a, Section 2] and [OSS14, Section 2].

Lemma 2.1. *Let $K = T_{4,5} \# -T_{2,3;2,5}$. Then $CFK^\infty(K)$ contains a direct summand generated over $\mathbb{F}[U, U^{-1}]$ by x, y , and z with*

$$\begin{array}{ll} M(x) = 0 & A(x) = 2 \\ M(y) = -3 & A(y) = 0 \\ M(z) = -4 & A(z) = -2 \end{array}$$

and differential

$$\partial x = 0 \quad \partial y = U^2 x + z \quad \partial z = 0.$$

Here, M and A denote the Maslov grading and Alexander filtration, respectively.

Proof. The knot $T_{2,3;2,5}$ is an L-space knot [Hed09, Theorem 1.10]; see also [Hom11b]. The Alexander polynomial of $T_{2,3;2,5}$ is

$$\begin{aligned} \Delta_{T_{2,3;2,5}}(t) &= \Delta_{T_{2,3}}(t^2) \cdot \Delta_{T_{2,5}} \\ &= t^4 - t^3 + 1 - t^{-3} + t^{-4}. \end{aligned}$$

Then by [OS05] (as restated in [OSS14, Theorem 2.10]), the complex $CFK^\infty(T_{2,3;2,5})$ is generated over $\mathbb{F}[U, U^{-1}]$ by a, b, c, d , and e with

$$\begin{array}{ll} M(a) = 0 & A(a) = 4 \\ M(b) = -1 & A(b) = 3 \\ M(c) = -2 & A(c) = 0 \\ M(b) = -7 & A(b) = -3 \\ M(c) = -8 & A(c) = -4 \end{array}$$

and differential

$$\partial a = \partial c = \partial e = 0 \quad \partial b = Ua + c \quad \partial d = U^3 c + e.$$

In the language of [HHN13, Section 2.4], we have that $CFK^\infty(T_{2,3;2,5})$ can be denoted $[1, 3]$, and the summand C specified in the statement of Lemma 2.1 can be denoted $[2]$. This notation refers to the lengths of the horizontal and vertical arrows in a graphical depiction of CFK^∞ , beginning from the generator of vertical homology and continuing to the point of symmetry. See Figures 1(a) and 1(b). It then follows from [HHN13, Lemma 3.1] that we have that $CFK^\infty(T_{2,3;2,5}) \otimes C$ is of the form $[1, 3, 2]$.

The Alexander polynomial of $T_{4,5}$ is

$$\Delta_{T_{4,5}}(t) = t^6 - t^5 + t^2 - 1 + t^{-2} - t^{-5} + t^6.$$

Since $T_{4,5}$ admits a lens space surgery, it is an L-space knot. Thus, we may apply [OSS14, Theorem 2.10] to obtain a description of $CFK^\infty(T_{4,5})$, and we see that, in the notation of [HHN13, Section 2.4], this complex is of the form $[1, 3, 2]$. See Figure 1(c).

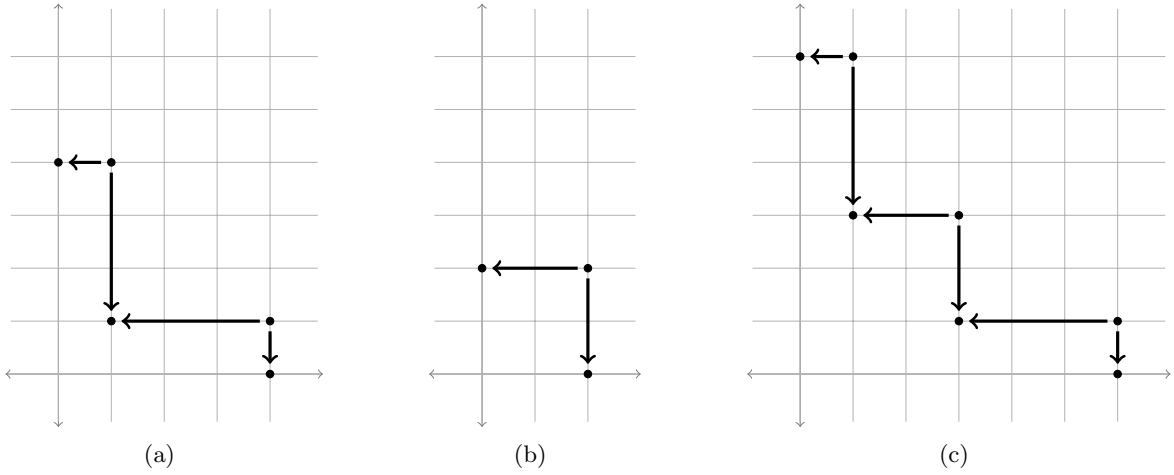


FIGURE 1. Left, $CFK^\infty(T_{2,3;2,5})$. Center, the relevant summand of $CFK^\infty(T_{4,5} \# -T_{2,3;2,5})$ from the statement of Lemma 2.1. Right, $CFK^\infty(T_{4,5})$. More precisely, CFK^∞ is generated over $\mathbb{F}[U, U^{-1}]$ by the generators depicted.

It follows from [HHN13, Section 2.4] that since $CFK^\infty(T_{2,3;2,5}) \otimes C$ has the same form as $CFK^\infty(T_{4,5})$, the complex C is a direct summand of $CFK^\infty(T_{4,5}) \otimes CFK^\infty(T_{2,3;2,5})^*$, or, equivalently, $CFK^\infty(T_{4,5} \# -T_{2,3;2,5})$. \square

Lemma 2.2. *Let $K = T_{4,5} \# -T_{2,3;2,5}$. Then*

$$\Upsilon_K(t) = \begin{cases} -2t & \text{if } 0 \leq t \leq 1 \\ 2t - 4 & \text{if } 1 < t \leq 2. \end{cases}$$

Proof. The summand of $CFK^\infty(K)$ described in Lemma 2.1 generates the homology of the total complex $CFK^\infty(K)$. In particular, this summand determines $\Upsilon_K(t)$. Although this summand is not itself CFK^∞ of an L-space knot [HW14, Corollary 9], the calculation in [OSS14, Proof of Theorem 6.2] still applies, yielding the desired result. \square

Lemma 2.3. *For the $(2, 5)$ -torus knot, we have*

$$\Upsilon_{T_{2,5}}(t) = \begin{cases} -2t & \text{if } 0 \leq t \leq 1 \\ 2t - 4 & \text{if } 1 < t \leq 2. \end{cases}$$

Proof. The result follows immediately from [OSS14, Theorem 1.15]. \square

With these lemmas in place, we are now ready to prove Theorem 2.

Proof of Theorem 2. By [OSS14, Propositions 1.8 and 1.9],

$$\Upsilon_{K_1 \# K_2}(t) = \Upsilon_{K_1}(t) + \Upsilon_{K_2}(t) \quad \text{and} \quad \Upsilon_{-K}(t) = -\Upsilon_K(t).$$

Combined with Lemmas 2.2 and 2.3, it follows that

$$\Upsilon_{T_{2,5} \# -T_{4,5} \# T_{2,3;2,5}}(t) \equiv 0.$$

We consider the invariant $a_1(K)$ defined in [Hom11a, Section 6]. For complexes such as the ones in Figure 1, the invariant $a_1(K)$ is equal to the length of the horizontal arrow coming in to the generator of vertical homology. From the partial description of $CFK^\infty(T_{4,5} \# -T_{2,3;2,5})$ in Lemma 2.1, it follows that

$$a_1(T_{4,5} \# -T_{2,3;2,5}) = 2.$$

By [Hom11a, Lemma 6.5] we have that

$$a_1(T_{2,5}) = 1.$$

Lastly, by [Hom11a, Lemma 6.3] we have that if $a_1(J) > a_1(K)$, then $\varepsilon(K \# -J) = 1$. Thus

$$\varepsilon(T_{2,5} \# -T_{4,5} \# T_{2,3;2,5}) = 1,$$

as desired.

Recall from [Hom14, Proposition 3.6] that for $n > 0$, we have

$$\varepsilon(nK) = \varepsilon(K) \quad \text{and} \quad \varepsilon(-K) = -\varepsilon(K),$$

It follows that any non-zero multiple nK of the knot $K = T_{2,5} \# -T_{4,5} \# T_{2,3;2,5}$ will also have the property that $\Upsilon_{nK}(t) \equiv 0$ and $\varepsilon(nK) \neq 0$. \square

Proof of Corollary 3. The invariant $\gamma(K)$ vanishes if and only if $\varepsilon(K) = 0$. Hence $K = T_{2,5} \# -T_{4,5} \# T_{2,3;2,5}$ (or any non-zero multiple thereof) has the desired property. \square

Remark 2.4. Let $K = T_{2,5} \# -T_{4,5} \# T_{2,3;2,5}$. By computing $CFK^\infty(K)$ using the Künneth formula [OS04, Theorem 7.1], one can determine that $\gamma(K) = 4$. More generally, we expect that $\gamma(nK) = 4n$, giving knots for which the concordance genus bound obtained from $\Upsilon_K(t)$ is zero, but the bound obtained from γ is arbitrarily large.

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